Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let p is positive integer such that $p \geq 2$ and let $b_n := \sum_{k=0}^{[\log_2 n]} \left[\frac{2^k + n}{2^{k+1}} \right]^p$.

Find $\min_{n \in \mathbb{N}} \frac{b_n}{n^p + 2^p - 2}$.

Solution.

$$b_n = \sum_{k=0}^{[\log_2 n]} \left[\frac{2^k + n}{2^{k+1}} \right]^p = \left[\frac{2^0 + n}{2^{0+1}} \right]^p + \sum_{k=1}^{[\log_2 n]} \left[\frac{2^k + n}{2^{k+1}} \right]^p = \left[\frac{n+1}{2} \right]^p + \sum_{k=1}^{[\log_2 n]} \left[\frac{2^k + n}{2^{k+1}} \right]^p.$$

But $\left[\frac{2^k + n}{2^{k+1}} \right] = \left[\frac{2^{k-1} + \frac{n}{2}}{2^k} \right] = \left[\frac{\left[2^{k-1} + \frac{n}{2} \right]}{2^k} \right] = \left[\frac{2^{k-1} + \left[\frac{n}{2} \right]}{2^k} \right]$.

Hence, $\sum_{k=1}^{[\log_2 n]} \left[\frac{2^k + n}{2^{k+1}} \right]^p = \sum_{k=1}^{[\log_2 n]} \left[\frac{2^{k-1} + \left[\frac{n}{2} \right]}{2^k} \right]^p = \sum_{k=0}^{[\log_2 n]-1} \left[\frac{2^{k-1} + \left[\frac{n}{2} \right]}{2^k} \right]^p$

and, since $[\log_2 n] - 1 = \left[\log_2 \frac{n}{2} \right] = \left[\log_2 \left[\frac{n}{2} \right] \right]$ we obtain $\sum_{k=1}^{[\log_2 n]} \left[\frac{2^p + n}{2^{p+1}} \right]^p =$

$\sum_{k=0}^{\left[\log_2 \left[\frac{n}{2} \right]\right]-1} \left[\frac{2^{k-1} + \left[\frac{n}{2} \right]}{2^k} \right]^p = b_{\left[\frac{n}{2} \right]}$ and, therefore,

(3) $\quad b_n = \left[\frac{n+1}{2} \right]^p + b_{\left[\frac{n}{2} \right]}, n \in \mathbb{N} \Leftrightarrow \begin{cases} b_{2k} = k^p + b_k \\ b_{2k+1} = (k+1)^p + b_k \end{cases}, k \in \mathbb{N}.$

For $n = 2^m, m \in \mathbb{N}$ we have $b_{2^m} = \left[\frac{2^m + 1}{2} \right]^p + b_{2^{m-1}} = 2^{p(m-1)} + b_{2^{m-1}}$.

Since $\sum_{i=1}^{m} 2^{p(i-1)} = \sum_{i=1}^{m} (b_{2^i} - b_{2^{i-1}}) \Leftrightarrow \frac{2^{pm} - 1}{2^p - 1} = b_{2^m} - b_{2^0} \Leftrightarrow$

$\frac{2^{pm} - 1}{2^p - 1} = b_{2^m} - 1 \Leftrightarrow b_{2^m} = \frac{2^{pm} + 2^p - 2}{2^p - 1} \Leftrightarrow b_n = \frac{n^p + 2^p - 2}{2^p - 1}$.

Further we will prove, using Math. Induction. that for any $n \in \mathbb{N}$ holds inequality

(4) $\quad b_n \geq \frac{n^p + 2^p - 2}{2^p - 1}$.

1. Base of Math. Induction.

For $n = 1$we have $b_1 = 1$ and $\frac{1^p + 2^p - 2}{2^p - 1} = 1$.

2. Step of Math. Induction.

For any $n > 1$ from supposition $b_k \geq \frac{k^2 + 2^p - 2}{2^p - 1}$, $k < n$ follow:

1. If $n = 2k$ then $b_n = b_{2k} = k^p + b_k \geq k^p + \frac{k^p + 2^p - 2}{2^p - 1} = \frac{(2k)^p + 2^p - 2}{2^p - 1} = \frac{n^p + 2^p - 2}{2^p - 1}$;
2. If $n = 2k+1$ then $b_{2k+1} = (k+1)^p + b_k \geq (k+1)^p + \frac{k^p + 2^p - 2}{2^p - 1} =$

$\frac{2^p(k+1)^p - (k+1)^p + k^p + 2^p - 2}{2^p - 1}$.

Remains to prove that $2^p(k+1)^p - (k+1)^p + k^p > (2k+1)^p$.

We have $2^p(k+1)^p - (k+1)^p + k^p > (2k+1)^p \Leftrightarrow$

$2^p(k+1)^p + k^p > (2k+1)^p + (k+1)^p \Leftrightarrow (2k+2)^p + k^p > (2k+1)^p + (k+1)^p \Leftrightarrow$

$(2k+2)^p-(2k+1)^p > (k+1)^p-k^p$, where latter inequality is right because function $h(x) = (x+1)^p-x^p = (x+1)^{p-1}+(x+1)^{p-2}x+\ldots+(x+1)x^{p-2}+x^{p-1}$ obviously is increasing in $(0,\infty)$.

We can see that equality in inequality $b_n \geq \frac{n^2+2}{3}$ occurs only if n is power of 2, because otherwise, in chain of inequalities at least one time appears rigorous inequality.

Thus, $\min\limits_{n\in\mathbb{N}} \frac{b_n}{n^p+2^p-2} = \frac{n^p+2^p-2}{2^p-1}$.